An action research on line to introduce fractals in the teaching and learning of mathematics from primary to secondary school

Adalberto Codetta Raiteri, OPPI, Milano, Italy, adalberto@codetta.it

0. Abstract

The times they are changin'. Not so the teaching and learning of mathematics. Maybe it's a characteristic of mathematics. Arabic numbers took many centuries to spread in Europe. Can the slowness of changes in the mathematics we teach at school influence the scientific orientation of students today? Can it restrict the vocation of new mathematicians? Which are the most effective means to experiment and divulge the teaching of new mathematical contents? In the CIEAM 55 conference it was suggested introducing the teaching of fractals[1] *in primary and secondary school. The author of this hypothesis have tried to experiment and divulge this proposal with the action research on line methods. The action research started in May 2004 and finished in June 2005. It involved 15 math teachers and8 teachers of other subjects, 15 classes and 12 schools. It was held without direct presence, entirely at a distance. We relate about the innovations introduced with the teaching of fractals and discuss the effectiveness and the eventual limits of a method that would allow operation on a broad scale, with very limited economical resources.*

1. Fractals why and how

Mandelbrot published "*Les object fractals*"[2] in 1975 and *"Fractals, Graphics & Mathematics education"*[3] in 2002. In these years the vision of science has changed deeply. It is not so for science as taught in schools. Action research intends overcoming an image of science still bound to determinism, in coherence with didactic action, in order to introduce the concepts of complexity, chaos and system. The research participants, divided into work groups of the same scholastic levels, make use of the resources available on the WEB, they plan and experiment in class suitable study itineraries for fractal objects, document the results of the experimentation, present the products made by students. Fractals need interdisciplinary approaches[4], they can be studied at different school levels to develop many mathematical topics: geometry of nature, self-similarity, logarithm, complex functions, recursive functions. For instance, students could discover the idea of self-similarity by directly exploring some fractals published on the Web, and, by schematising them, they could arrive at the development of some simple mathematical models of self-similarity like the curve of Koch.

There are many reasons for using fractals as didactic material from elementary to secondary schools:

- the actuality of fractal models which are used in many applications, from medicine to cinematography
- the aesthetical aspects which also involve the students emotional intelligence
- wide bibliography published on the web especially with didactical purposes
- the aid of computers allows the student to handle curves and concepts previously reserved to mathematicians and allows the teacher a lower use of technicalities in the program
- the nature of fractals emphasizes the perception that even in mathematics we "invent" rather than "discover"

Is it really necessary to make students cover every step that humanity has made to achieve certain concepts? This research assumes, as a hypothesis, that it is not always necessary. It intends verifying if it is possible to get teachers to plan suitable ways to present at least some aspects of current science, also to the younger students. The motivation of the young towards scientific studies is at stake and also the weight of scientific formation in the new generations. Fractals are not present in educational programs but many contents in the programs allow a description of some of the characteristics of fractal objects. The research aims at identifying didactic channels that allow students to get used to fractal objects linking them to mathematical concepts pertaining to educational programs.

2. Definitions of action research on line

We accept the following definition of action research: *"Action research can be described as a family of research methodologies which pursue action (or change) and research (or understanding) at the same time. In most of its forms it does this by using a cyclic or spiral process which alternates between action and critical reflection and in the later cycles, continuously refining methods, data and interpretation in the light of the understanding developed in the earlier cycles. It is thus an* ***emergent*** *process which takes shape as understanding increases; it is an* ***iterative*** *process which converges towards a better understanding of what happens. In most of its forms it is also participative (among other reasons, change is usually easier to achieve when those affected by the change are involved) and qualitative"* [5]. The action research can have a large variety of typologies. We have tried to organise a pilot research: "*The pilot research tends to explore a predefined sphere125% in function of an acquired status of research, in a domain where one begins to see interesting dimensions, but in which strategies, guidelines etc... have not yet been set, or one feels the need to highlight more specific dimensions or hypotheses. A focus of investigation and a certain number of restrictions, therefore, exist (depending on the typologies already defined) but autonomous spaces for exploration also remain for the actors. The aim is above all to deduce from this research operative typologies or repertories that can be transferred to other situations"*[6]

3 The climate of research

The participants in the research live all over Italy. Face to face meetings are not foreseen. This means they cannot make use of the thousands of pieces of information coming from the context, gestures, tone of voice, mime. It is therefore useful to imagine the context in which everyone participates in the research. We are more and more urged to a frenetic life. The potential of access to internet increases the anxiety of participation, presence, extension of knowledge. Teachers, because they don't have a clear cut between working and personal hours, are particularly exposed to the global pressure of the net and to the conflict with those of their children, spouses, friends, … "real" life. Internet does not make learning faster. The excessive load of sources of information, links, people involved requires exacting metacognition operations to construct stable and well-organised personal learning. Distance learning, on line, on the web requires detachment, calm, serenity, reflection. This applies for students and for adults. Nevertheless, e-learning platforms allow the advantages of asynchronous learning: one need not to be physically present at meetings, sacrificing personal and professional, engagements, one can choose the best moment to dedicate oneself to the action research, to the comparison with colleagues, to reflection and to studying with the sole aim of a personal and professional growth.

4. Chronicle of the research action

The on line research started on the 21st May 2004. Following brief communications on the web with an invitation to take part in a research on fractals, 74 teachers showed interest. Each one received a personal code for access to the work platform on-line. On the on-line platform the teachers found the base document for the research, a possible work model, numerous links to qualified sites that deal with fractals and the opportunity to compare by way of contributor forums and archives. The base document asked them to present a project, based on a shared model, to experiment with it in class and present the results obtained. The research coordinator took on the responsibility of certifying the work done and giving assistance. Always done through distance work. The proposal was only accepted by a limited number of teachers who presented a project. Only 23 teachers have been enrolled (by their headmasters) and have presented 15 projects (1 in primary school, 4 in 1st grade secondary school, 4 in the two-year course of 2nd grade secondary school, 4 in the three-year course of 2nd grade secondary school). In some of the projects a number of teachers from the same school are taking part. Two types of products emerge from the experimentation in class:

- Products made by teachers for students
- Products made by students during or at the end of the study itinerary of fractals

In order to allow a shared interpretation, at the end of the study itinerary on fractals, every involved student will receive a questionnaire that will be placed at disposal on the work platform. The questionnaire will focus on the main question "*what is a fractal according to you*?"[7]

For every project, the participant teachers presented a report. Attached to the report, there are products made by the teachers, products made by the students, exemplary questionnaires. These documents will allow an evaluation of the work done.

5. An instrument of analysis

In order to compare the strategies of the teachers, we asked them to adopt the same stile of documentation. So we can transfer many key sentences describing each experimentation to a table of comparison. This table is organized age by age in order to recognize some emergence.

5.1 Experimentations in classes with students 9 to 14 years old

The teachers with their students explored a large variety of meanings but a careful analysis of the table shows that in 9-11 the focus is on geometrical transformation while from 12-14 the focus moves to iterative processes.

Teacher's aims	Student objectives	Didactic path and topics
9-10 years old students / Italian primary school, fourth years		
Contribute in the formation of a scientific thought which has to be flexible and open. Make the pupils understand that many models and many "mathematics" are formulated in order to represent reality.	Understand properties of triangles in Euclidean geometry Understand concepts of translation, rotation, axial and central symmetry, omothetia, similitude, Construct fractals : Koch snowflake, triangle Sierpinsky Understand the meaning of selfsimilarity	Animated examples for the construction of fractals, explained and commented by the use of rhymes. Studies of triangles in Euclidean geometry. Observation of Sierpinsk triangle and the most famous curves such as cardioid, Peano, Kock. Construction of fractal object. Comprehension of the concept of self similarity Materials to be used: rhymes which describe the construction process of a fractals, electronic animation, cognitive maps which represent student's learning process, software Logo and Fractint
11-12 years old students / Italian first grade of secondary school, first years		
Start the process of mathematization concerning physical objects: observing, formulating questions, seeking answers. Stimulate the student's ability to decompose a problem into sub-questions and to organize observations in logic sequences in order to afford complex problems.	Understand the concept of shape. Search for regularities. Recognize geometrical figure from the properties which characterize them Recognize the invariant properties within a transformation Acquire a correct language in order to refer about carried out experiences	Visualization of tridimensional object from bidimensional representation: plane section., Introduction to topology, open line, close line, connected line: Mobius strip, Peano curves . geometrical transformation: axial simmetry and rotation. Concepts of shape and self similarity . Iterative process: manipulation using paper, use of software. Experience of geometry in nature observing: leaves disposition, representation of a leaf considering its invariant properties. Representative model of a leaf: the fern Observation and manipulation activities, web navigation, software Cabri

12-13 years-old students/ Italian first grade secondary school, second year		
Stir up interest and motivations in the students in order to increase and develope intuitive and creative abilities. To excite observing ability concerning facts and phenomenon of reality	Grasp analogies and differences, both variant and invariant. Afford complex problems decomposing them into sub-question. Use the concept of measure within different contest. Use language and technologies from computer science. Use conceptual maps See the artistic aesthetic side of a geometrical figure	Mathematical modelling: observation of regular figure aside from reality (fern leaf) .Modelling within the Euclidean geometry. Rise of fractal geometry as necessary model in order to observe and represent reality. Search of fractals figures: presentation using software Summary about SEW-COM method for web research. Search on the internet of web-sites about fractals Concepts of dimensions, perimeter, area: perimeter of some plane figures, perimeter of a seabord. Construction, using CABRI, of a tree and a snowflake. Analysis of fractal figure done by artists. Use of software (FRACTIN, FRACTAL EXPLORER)
13-14 years-old student's / Italian first grade secondary school, third year		
Stimulate student's ability to afford more complex problem decomposing them into sub-questions. Induce students to understand and use specific languages within the fields of tecnic, science and multimedia. Guide the students into data and informations selection in order to a given end.	Grasp the fundamental characteristic of any fractal: self similarity. Describe the modular structure of a fractal. Connect recursive sequences to recursive algorithms. Carry out the basic structure of a fractals using CABRI software. Draw simple fractals through recursive algorithm using macro of CABRI. Understand fractals as interpretative model of reality. Analyse perimeter and area variations of some fractal figures. Understand and calculate dimensions of a fractal curve.	Historical study: the problem of "irregular" shapes in reality which can't be described by classic geometry and the search for new theories. Discovery of the main characteristic of these irregular shapes: they are made up of repetitive structures which can be seen, described and reproduced . Guided search of fractals in the web. Presentation of fractals as interpretative model for nature which explain reality through algorithm. Observation and realization, using CABRI, of fractals curve. Introduction about the concept of fractal dimension as rational number, comparison with classic figure dimension as integer. Search, using EXCEL, of fractal dimension of realized curves. Search of fractal figure in modern and contemporary art collaborating with arts teacher.

5.2 Experimentations in classes with students 14 to 19 years old

The teachers adapted topics connected with fractals to the goals of their different schools. Fractals can be easily adapted to cover a large variety of skills as shown by comparing experimentations in vocational, art, science and technical schools. The focus of activities for students from 14 to 19 years old students, gradually moves from the knowledge of geometrical transformations and iterating processes to their implementation in a programming language. Cultural and scientific aspects of fractals are taken in consideration. The very high numbers of WEB sites which propose different and creative approaches to the study of fractals suggested to teachers to join fractals study with some reflections about methods of research on the WEB.

Teacher's aims	**Student objectives**	**Didactic path and topics**
14-15 years old students / Italian vocational school, first year		
Use ICT to experiment personalised didactic path. Stimulate the development of abilities to use software on geometry connected to capacities to describe and reflect experiences. Develop cooperative learning styles through problem solving method.	Develop spatial intuition Deepen geometrical basic knowledge.	See again, through CABRI, the concept of triangle, polygon, perimeter, area, ….. Use and comprehend the macro in CABRI Understand the concept of geometrical transformation Study and describe the characteristic of snowflake and Sierpinsky triangle. Create some fractal shape.

14-15 years old students / Italian art school, first year		
Present a live idea of mathematics. Use ICT to make students protagonist of their geometry learning Promote interdisciplinary approach connecting mathematics and art. Promote understanding and comparison of languages used in different fields of knowledge	Use ICT resources to study, generate, visualise fractal objects Select informations on the WEB through SEWCOM method Connect different information through appropriate models of interpretation Organise and represent the acquired knowledge through concept maps and nets of maps.	Characteristics of fractals objects and related math topics: self-similarity, geometrical transformations, algorithms, iterative functions, fractals attractors. Fields of knowledge which uses fractals Some famous fractal object: Mandelbrot set, Sierpinsky net History of fractal and authors Fractal and art, nature and technology
16-17 years old students/Italian science school, third year		
Promote the ability to find information in the ICT context Make students aware of the different instruments and methods to search information	Know and use SEWCOM method to select informations Be aware of the importance of fractal as a model of science phenomenon Insert the theory of fractals in the actual development of mathematical research	History of fractals Practical applications of fractals Study of famous fractal objects Construction with Cabri of triangle of Sierpinsky and Snowflake Computation with Pascal of the area of triangles of Sierpinsky ant the perimeters of Snowflake
17-18 years old students/ Italian science school, fourth year		
Apply the usual mathematics program in a new field which can fascinate students Promote the ability to select and find the information on the WEB Develop an auntomous style of working	Use concept maps and nets of maps to describe and reorganise the knowledge acquired during the school research Use ICT resources to find and communicate information Know and use mathematical models in different contexts	Discovery of self similarity through analysis of fractals with Tierazon. Measurement of the length of a cost, fractal dimension. Geometrical transformations: study of Sierpinsky, Kock and other IFS fractals. Recursive functions: design of a fractal figure using Pascal Logarithm: dimension of fractal object Complex numbers: study of Mandelbrot set and Julia fractals Probabilty: non deterministic fractals Applications of fractal geometry
17-18 years old students/ Italian technical school, fourth year		
Improve the attitude to make a critical analysis on the previous acquired knowledges Develop practical skill of mathematical modelling	Comprehend the characteristic of fractal geometry, the differences and invariances with euclidean geometry Break problems into several smaller subproblems	Project fractals trough recursive procedures distinguishing IFS from LS fractal Create fractals with different programming languages
18-19 years old students/ Italian technical school, fifth year		
Present an idea of mathematics which overcomes old problems using actual and real problems Develop skills to evaluate different programming languages in order to solve different problems	Comprehend characteristics of fractal geometry, differences and invariances with euclidean geometry Analyse useful and useless applications of fractals Know and use the SECOM method to find and select informations	Triangle of Sierpinsky, Teory of fractald IFS, Fractals LS, fractal dimension Create fractals with Pascal, C++, Java, Logo, Cabri, Excel, distinguishing IFS from ILS ones, Design of a WEB site on fractals

6. Conclusions

The teachers produced a large variety of didactic materials on fractals: lessons, guides, problems, concept map, questionnaires, didactic software, tests, ...even rhymes! The students also contributed to this abundance: hand drawings, computer drawings, geometrical constructions, articles, hypertexts, computer programs, web sites, web references.... These materials, because of their richness and complexity, are not jet completely examined and classified. A deeper analysis could give many indications in order to introduce fractals in any level of school and to project teachers training courses. At the actual level, this on-line research can address the following statement:

- Fractals can be studied at any level of school
- Fractals give many opportunities to make "good mathematics"
- Mathematics models of fractals enjoy students and push some of them to deepen more math topics
- Fractals can be studied with heuristic approaches starting from reality problems
- Fractals give to mathematics teachers an important role in interdisciplinary activities which are actually more and more requested by school systems
- Fractals studies needs WEB interactions and working group. In this way, fractal studies foster behaviour that helps active research in both teachers and students
- Fractals can be studied within the ordinary mathematics programs: geometrical transformations, infinite sequences, infinite series, recursive functions, logarithm, complex numbers, probability...

This research has been held completely on line. The teachers listed below worked together for more than one year without meeting each other This was e new working environment for a large part of the teachers involved Difficulties of this new context were profitably overcome. The results of this research show that the cooperative work supported by e-learning platform (Claroline, manager Doriano Azzena, host, IPSIA Castigliano Asti) opens interesting opportunities to develop teacher's professional roles. Primo Brandi and Anna Salvatori, "Progetto Innovamatica", Perugia University, Engineering Department, supported the research giving assistance and advice. They participate to the scientific committee of the research with Stefania Marangoni, Renza Cambini and Laura Lotti. The author thanks them, the teachers and the directors of schools participant.

7. Teachers, classes, schools participant

teacher	subject	students age	School
Ivana Niccolai	Math , science	8-10	Scuola Elementare "G. Garibaldi", Genova
Letizia Corniani	Math, science	11-12	Istituto Comprensivo I Suzzara (MN)
Ernestina Prada Gianfranco Damiano Chiara Maggioni	Math, science Italian Art education	12-13	Istituto Comprensivo Barlassina (MI)
Susanna Abbati Rosella Ghezzi	Math, science Art education	12-13	Istituto Comprensivo "Rodari", Baranzate, MI
Gianpaolo Maran Vincenzo Trabona	Math, science Art education	12.13	Istituto Comprensivo 7, Vicenza
Mariarosa Sanfelici	Math, science	12-13 13-14	Scuola Media "B. Croce" Gonzaga, MN
Marzia Galafassi Carla Tabai	Math, ICT Math, ICT	14-15 14-15	Istituto d'istruzione Superiore "S. G. Bosco" Viadana (MN)
Luca Vampa Renata Casagrande Giovanna da Col	Math Painting Geometry	14-15	Istituto Statale d'Arte "Bruno Munari", Vittorio Veneto (TV)
Adriana Minocci	Math	16-17	Liceo Scientif. "G. Spezia",Domodossola (VB)
Anna Venditelli Gianluca Tiengo	Math Math lab.	17-18	Istituto Tecnico Industriale "E.Majorana" Cassino (FR)
Marina Celora Antonella Montrezza Carmen Giovanelli Vitaliano Caimi	Math, Physics Math, Physics Science Philosophy	17-18 17-18	Liceo Scientifico "A. Tosi" Busto Arsizio (VA)
Antonella Trevisol	Math	18-19	ITIS "Cartesio" Cinisello Balsamo (MI)

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Laura Lotti; http://www.frattali.it/

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Giorgio Pietrocola: http://www.maecla.it/tartapelago.htm

10. Software

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Flarium 24; http://ktaza.com/fractal/

Fractal Explorer; http://www.eclectasy.com/Fractal-Explorer/